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A Variational Approach to Low Peclet Number Heat Transfer in Laminar Flow*

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The boundary method of Galerkin is used to solve the problem of heat transfer in laminar flow with axial conduction. The set of particular solutions used in this calculation consists of the product of exponentially decaying functions (in the positive axial direction) and radially dependent confluent hypergeometric functions of Kummer's type. Nusselt functions and temperature profiles are presented and comparison is made with respect to the number of terms used in the trial function. The same problem is also treated by the method of boundary collocation with subsequent comparisons with the boundary method of Galerkin. Computationally, the methods presented here appear to offer considerable advantages over previously employed methods such as the interior method of Galerkin.

INTRODUCTION

The linear, elliptic boundary value problem governing heat transfer in laminar flow has been solved approximately by two different methods: finite difference techniques [1, 2] and variational methods [3–7]. In [1] McMordie and Emery used a forward-differencing technique to solve the problem of laminar-flow heat transfer with axial conduction and developing velocity profile, while in [2] Hennecke employed a central-differencing technique to solve the problem of laminar-flow heat transfer with axial conduction, fully developed velocity profile, and adiabatic entrance conditions. Millsaps and Pohlhausen [3] and Singh [4] treated the problem of constant entrance temperature, fully developed laminar flow, and axial conduction. Employing the interior method of Galerkin, these authors used a trial solution of the form

$$T(r, z) = \sum_{n=1}^{N} A_n T_n(r) \exp(-\lambda_n z),$$

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where the radially dependent functions $T_n(r)$ were expanded as infinite series of zero-order Bessel functions with expansion coefficients $B_m^{(n)}$. The resulting equations for the discrete values assigned to the λ_n involved determinants of infinite order, and the difficulty of computing λ_n increased considerably as n became larger. After each of the first N of the λ_n were computed, $(N-1)$ of the corresponding expansion coefficients $B_m^{(n)}$ had to be obtained. Finally, after considerable effort, a least-squares procedure could be used to obtain the coefficients A_n.

A shorter approach was employed by Hsu [6] who solved for the λ_n and $T_n(r)$ directly by the Runge–Kutta method. In his first paper, Hsu used an incorrect formula derived by Singh [8] (see also [9, 10]) to compute the expansion coefficients A_n. In the same paper, he also used an asymptotic relation for $T(r, z)$ which was incompatible with his constant entrance temperature assumption. In his second paper, Hsu considered the case of an adiabatic entrance condition [7]. Here, although he mistakenly thought of his method as exact, Hsu actually used the boundary method of Galerkin to compute the coefficients A_n. Furthermore, Hsu's use of the numerically unstable Gram–Schmidt procedure [11, 12] to solve the equations resulting from the Galerkin method might have led to the incorrect values of the A_n he obtained for low Peclet numbers. He had to correct these values by an iteration procedure.

The main purpose of the present paper is to present a more efficient procedure for the calculation of the λ_n and $T_n(r)$ of the foregoing trial solution and also to present results of various variational methods for calculating the expansion coefficients A_n. This procedure avoids the lengthy interior method of Galerkin by utilizing the exact functional form for $T_n(r)$; thus, the more direct boundary method of Galerkin or the boundary collocation method can be employed to determine the A_n. A secondary purpose is to present the results obtained for low Peclet numbers for the problem of constant entrance temperature, with a compatible asymptotic temperature relation. Physically this problem corresponds to a viscous liquid entering a cylindrical capillary at low velocity from a well-stirred, constant temperature reservoir. Reynolds numbers of approximately unity can result from this situation, leading to 99% of the velocity profile development in less than 0.06 capillary diameters [13]. In this case the Peclet number can still be reasonably low (about 10 or 20) even when the Prandtl number is about 10. Thus the validity of this model is not restricted to liquid metals.

The Energy Equation

In addition to assuming instantaneous velocity profile development, constant reservoir temperature T_0, axial and radial conduction, and steady state conditions, the particular problem considered here will assume constant tube radius r_0,

parabolic flow, constant thermal conductivity k, density and heat capacity, negligible heat dissipation by viscous forces, and constant heat flux q through the tube wall. The energy equation, derived in the usual manner by performing a differential balance of convective and conductive heat fluxes, then reduces to the elliptic partial differential equation:

$$(1 - \xi^2)\,\partial\tau/\partial\eta = (1/Pe)(\partial^2\tau/\partial\xi^2 + \xi^{-1}\,\partial\tau/\partial\xi + \partial^2\tau/\partial\eta^2) \tag{1}$$

$$\tau(0, \xi) = 0 \tag{2}$$

$$\tau \to \tau_f(\eta, \xi) \tag{3}$$

$$\tau \text{ finite at } \xi = 0 \tag{4}$$

$$\partial\tau/\partial\xi = 1, \text{ at } \xi = 1. \tag{5}$$

The dimensionless variables given in Eqs. (1)–(5) are defined by:

$$\tau = k(T - T_0)/(qr_0), \qquad \xi = r/r_0\,, \qquad \eta = x/r_0\,.$$

Here, r and x are the axial and radial coordinates, respectively, T is the temperature, and Pe denotes Peclet number.

The complete solution for the dimensionless temperature profile will be of the form

$$\tau(\eta, \xi) = \tau_f(\eta, \xi) + \theta(\eta, \xi) \tag{6}$$

in which $\theta(\eta, \xi)$ decays exponentially with η, resulting in $\tau(\eta, \xi) \sim \tau_f(\eta, \xi)$ for large η.

Asymptotic Solution to Energy Equation

The asymptotic solution τ_f must satisfy Eqs. (1), (3)–(5) but does not satisfy Eq. (2) since $\tau \to \tau_f$ only for large values of η. Hence, another boundary condition is needed to fully determine τ_f. We obtain this boundary condition, after [14], by performing a heat balance over the tube from $x = 0$ to $x = x'$ and then taking the limit as $x' \to \infty$. The result is

$$\tau_f = (4Pe^{-1})\,\eta + \xi^2 - \xi^4/4 - 7/24 - (4Pe^{-1}) \int_0^1 [\partial\theta(0, \xi)/\partial\eta]\,\xi\,d\xi. \tag{7}$$

Solution of the Function θ

We now obtain the solution θ which satisfies Eq. (1) and, because of the boundary conditions satisfied by τ_f and τ, must also satisfy certain boundary conditions of its own: θ finite at $\xi = 0$, $\theta \to 0$ as $\eta \to \infty$, $d\theta/d\xi = 0$ at $\xi = 1$, and

$$\theta(0, \xi) = -\tau_f(0, \xi) = -\xi^2 + \xi^4/4 + 7/24 + (4/Pe) \int_0^1 \frac{\partial\theta(0, \xi)}{\partial\eta} \xi \, d\xi. \quad (8)$$

To find the solution θ we begin with a procedure used by Walker [15], Passell and Perry [16], and Hsu [6]. Assume $\theta(\eta, \xi)$ has the form

$$\theta(\eta, \xi) = \sum_{n=1}^{N} C_n R_n(\xi) \exp(-\phi_n^2 \eta/Pe) \quad (9)$$

which already satisfies the condition $\theta \to 0$ as $\eta \to \infty$. Substituting Eq. (9) into Eq. (1) we obtain an ordinary differential equation for $R_n(\xi)$:

$$d^2R_n/d\xi^2 + \xi^{-1} \, dR_n/d\xi + [\phi_n^4/Pe^2 + \phi_n^2(1 - \xi^2)] \, R_n = 0.$$

The solution to this equation which satisfies the boundary condition $R_n(0)$ finite is

$$R_n(\xi, \phi_n) = \exp(-\phi_n\xi^2/2) \, M(a_n \, ; 1; \phi_n\xi^2), \quad (10)$$

where $M(a; b; y)$ is Kummer's confluent hypergeometric function [17], and

$$a_n = \tfrac{1}{2} - (\phi_n^3 + Pe^2\phi_n)/4Pe^2. \quad (11)$$

The discrete values of ϕ_n required for the necessary boundary condition $R_n'(1) = 0$ to be satisfied are obtained as the roots of

$$(a_n + \phi_n/2) \, M(a_n \, ; 1; \phi_n) - a_n M(1 + a_n \, ; 1; \phi_n) = 0. \quad (12)$$

However, since the negative roots of (12) are just the negatives of the positive roots and since $R_n(\xi, \phi_n)$ equals $R_n(\xi, -\phi_n)$ [17], we need only determine the positive roots of (12). (The ϕ_n depend on the Peclet number only; and the left-hand side of Eq. (12) can be tabulated versus a sequence of closely spaced values of ϕ_n. Taking as first estimates those values of ϕ_n between which the sign of the left-hand side changes, the method of false position was used to get more refined values of ϕ_n. For $Pe = 5$ and 10 the first 25 values of ϕ_n are given in Table I.)

TABLE I

Eigenvalues φ_n Corresponding to $Pe = 5$ and 10

	φ_n	
n	$Pe = 5$	$Pe = 10$
1	3.5988876	4.3345060
2	5.2843136	6.7407717
3	6.5834339	8.6329181
4	7.6746650	10.229411
5	8.6323615	11.629107
6	9.4954903	12.887527
7	10.287275	14.038978
8	11.022817	15.106130
9	11.712565	16.104778
10	12.364101	17.046403
11	12.983125	17.939641
12	13.574053	18.791169
13	14.140385	19.606272
14	14.684951	20.389214
15	15.210078	21.143494
16	15.717706	21.872022
17	16.209474	22.577251
18	16.686779	23.261270
19	17.150825	23.925872
20	17.602658	24.572615
21	18.043194	25.202859
22	18.473238	25.817801
23	18.893506	26.418498
24	19.304635	27.005895
25	19.707196	27.580833

In evaluating Kummer's function $M(a; b; x)$, its defining series [17] is used for $-1 \leqslant a \leqslant 1$. For $a \leqslant -1$, the defining series is used in conjunction with the recurrence relation

$$(b - a)\, M(a - 1; b; x) + (2a - b + x)\, M(a; b; x) - aM(a + 1; b; x) = 0. \quad (13)$$

Computationally, calculating the ϕ_n and $R_n(\xi, \phi_n)$ in this manner appeared to require less than a third of the computer time required in the Runge–Kutta procedure used by Hsu [6].

To obtain the coefficients C_n in Eq. (9), we substitute Eq. (9) into Eq. (8) to get

$$\sum_{n=1}^{N} C_n R_n(\xi) = -\xi^2 + \xi^4/4 + 7/24 - 4Pe^{-2} \sum_{n=1}^{N} \phi_n{}^2 C_n \int_0^1 y R_n(y, \phi_n)\, dy. \quad (14)$$

Equation (14) can be rewritten as

$$\sum_{n=1}^{N} C_n g_n(\xi) = -\xi^2 + \xi^4/4 + 7/24, \quad (15)$$

where

$$g_n(\xi) = R_n(\xi, \phi_n) + 4Pe^{-2}\phi_n{}^2 \int_0^1 y R_n(y, \phi_n)\, dy. \quad (16)$$

Because we are not dealing with a Sturm–Liouville system, we do not know a simple orthogonality relation, if any, to employ in evaluating the C_n. Thus, we are forced to use other techniques to determine the coefficients C_n.

Two variational techniques, the boundary method of Galerkin and the least-squares approximation, both reduce to the system of equations

$$\sum_{n=1}^{N} C_n(g_n, g_m) = (f, g_m), \qquad m = 1, 2, \ldots, N, \quad (17)$$

where

$$(u, v) \equiv \int_0^1 w(\xi)\, u(\xi)\, v(\xi)\, d\xi. \quad (18)$$

A Gaussian elimination method can then be used to obtain the coefficients C_n from Eq. (17), for a given weighting function w.

Another approach is to preorthonormalize the functions $g_n(\xi)$ in Eq. (15) via a modified Gram–Schmidt procedure. (The ordinary Gram–Schmidt orthogonalization procedure tends to be numerically unstable; the modification used

in the computation given in this paper has been devised to give better numerical results [12].) The preorthonormalization procedure (see [18] for details) results in a system of N linear nonhomogeneous equations which can be represented in matrix form as $BC = L$. The $N \times N$ matrix B is upper triagonal; thus the coefficient matrix C is easily obtained.

Hsu [7] used the preorthonormalization approach in the problem of constant heat flux to a liquid metal in laminar flow with an adiabatic entrance condition. He appeared to have used the ordinary Gram–Schmidt orthonormalization procedure which may have led to the computational difficulties he experienced in computing the coefficients C_n for the lower Peclet numbers. At any rate, the preorthonormalization method of obtaining the C_n did not prove advantageous (from a standpoint of computer time) to Gaussian elimination.

Another method that can be used to obtain the coefficients in (15) is the method of collocation [19]. This technique involves forcing the two sides of (15) to be equal at a number of points $\xi_1, \xi_2, \ldots, \xi_N$, resulting in the system of N linear algebraic equations:

$$\sum_{n=1}^{N} C_n g_n(\xi_k) = -\xi_k^2 + \xi_k^4/4 + 7/24, \qquad k = 1, \ldots, N. \tag{19}$$

Temperature Profile and Nusselt Number

After the C_n are obtained, substitution into Eq. (9) yields the approximation to θ. The expression for τ then becomes, when Eqs. (7) and (9) are substituted into Eq. (6),

$$\tau(\eta, \xi) = 4\eta/Pe + \xi^2 - \xi^2/4 - 7/24 + \sum_{n=1}^{N} C_n e^{-\phi_n^2 \eta/Pe} R_n(\xi)$$

$$+ \frac{4}{Pe^2} \sum_{n=1}^{N} C_n \phi_n^2 \int_0^1 y R_n \, dy. \tag{20}$$

Following Hsu [6], the expression for the Nusselt number becomes

$$Nu = 2\left[11/24 + \sum_{n=1}^{N} C_n \left(R_n(1)\, e^{-\phi_n^2 \eta/Pe} + \frac{4}{Pe^2} \phi_n^2 \int \xi R_n e^{-\phi_n^2 \eta/Pe} \, d\xi\right)\right]^{-1}. \tag{21}$$

Superficially, this is the same expression as obtained by Hsu [6], but the coefficients C_n reported by Hsu are not the same, due to his incorrect asymptotic boundary condition.

Computational Procedure

When using the boundary method of Galerkin to compute the coefficients C_n of Eq. (15), both Gaussian elimination and the preorthonormalization technique (using the modified Gram–Schmidt procedure) were carried out for $Pe = 10$ and $N = 21$. Double precision arithmetic was utilized throughout the calculations, and the two techniques gave identical values of C_n to at least ten significant figures. The Gaussian elimination technique required 30% less computer time than the preorthonormalization procedure. All integrations involved in the Galerkin method were performed with Simpson's rule using an integration interval of $\Delta\xi = .01$. Reduction of this interval to $\Delta\xi = .005$ changed the values of the C_n from only .001% for C_1 up to 2% for C_{25}.

The Gaussian elimination method used was a Fortran IV subroutine from the APL Computing Center library. This routine utilized double-precision arithmetic and complete pivoting, and was designed to report possible loss of significance at each elimination step. No such losses were reported for any of the calculations reported here.

The weighting function appearing in Eq. (18) and also used in the preorthonormalization method was arbitrarily chosen to be $w(\xi) = \xi(1 - \xi^2)$ since this is the natural weighting function for the case of no axial conduction.

Double precision arithmetic and the Gaussian elimination technique described above were also used in the collocation method for determining the C_n.

Results

A. *Boundary Method of Galerkin*

The coefficients of Eq. (15) computed by the boundary method of Galerkin for $Pe = 10$ and $N = 4, 12, 18, 21$, and 25 are shown in Table II. The lower coefficients (corresponding to $n < N/2$) change slightly as N increases, but the higher coefficients change more rapidly.

Using these coefficients, two quantities were computed: the Nusselt number of Eq. (21) and the quantity

$$F_1 = \int_0^1 \left[\sum_{n=1}^{N} C_n g_n - (\xi^4/4 - \xi^2 + 7/24)\right]^2 \xi(1 - \xi^2)\, d\xi. \tag{22}$$

The Nusselt numbers (actually the Nusselt functions since they depend on η) are shown in Table III for $N = 4, 12, 18, 21$, and 25. As can be observed, the convergence with respect to N is rapid for $\eta \geq 0.2$, and twelve terms appear sufficient

TABLE II

Expansion Coefficients Obtained by Method of Galerkin for $Pe = 10$

	C_n				
n	$N = 4$	$N = 12$	$N = 18$	$N = 21$	$N = 25$
1	0.440341	0.472379	0.472964	0.473090	0.473196
2	−0.171434	−0.215712	−0.216651	−0.216851	−0.217021
3	0.103028	0.121231	0.122484	0.122750	0.122976
4	−0.050039	−0.075292	−0.076825	−0.077149	−0.077423
5		0.050339	0.052136	0.052511	0.052829
6		−0.035514	−0.037573	−0.037998	−0.038355
7		0.025986	0.028316	0.028789	0.029184
8		−0.019423	−0.022052	−0.022571	−0.023004
9		0.014614	0.017585	0.018154	0.018624
10		−0.010868	−0.014261	−0.014882	−0.015390
11		0.007734	0.011695	0.012372	0.012920
12		−0.004782	−0.009650	−0.010390	−0.010980
13			0.007972	0.008783	0.009418
14			−0.006553	−0.007449	−0.008134
15			0.005312	0.006311	0.007052
16			−0.004216	−0.005361	−0.006173
17			0.003177	0.004525	0.005419
18			−0.002041	−0.003734	−0.004723
19				0.002998	0.004117
20				−0.002274	−0.003582
21				0.001480	0.003126
22					−0.002605
23					0.002002
24					−0.001501
25					0.001005

TABLE III

Nusselt Function vs. Number of Terms Used in Galerkin Method for $Pe = 10$

	Nu				
η	$N = 4$	$N = 12$	$N = 18$	$N = 21$	$N = 25$
0.0	18.825	75.002	112.760	131.647	157.565
0.2	7.745	8.474	8.494	8.498	8.502
0.4	6.007	6.218	6.223	6.224	6.224
0.6	5.316	5.412	5.414	5.415	5.415
0.8	4.958	5.010	5.011	5.011	5.011
1.0	4.749	4.780	4.780	4.781	4.781
2.0	4.619	4.421	4.421	4.421	4.421
∞	4.364	4.364	4.364	4.364	4.364

TABLE IV

Least-Square Parameter F_1 vs. Number of Terms Used in Galerkin Method for $Pe = 10$

N	$F_1 = \int_0^{'} \left[\sum_{n=1}^{N} C_n g_n - (\xi^4/4 - \xi^2 + 7/24)\right]^2 \xi(1 - \xi^2)\, d\xi$
4	5.101×10^{-5}
12	7.458×10^{-8}
18	2.293×10^{-8}
21	0.553×10^{-8}
25	0.217×10^{-8}

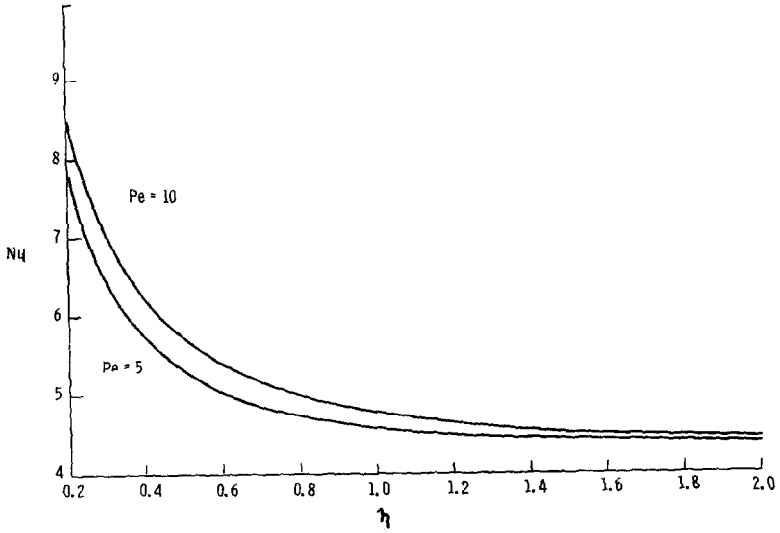

FIG. 1. Nusselt function obtained by Galerkin method with $N = 21$ and $Pe = 5, 10$.

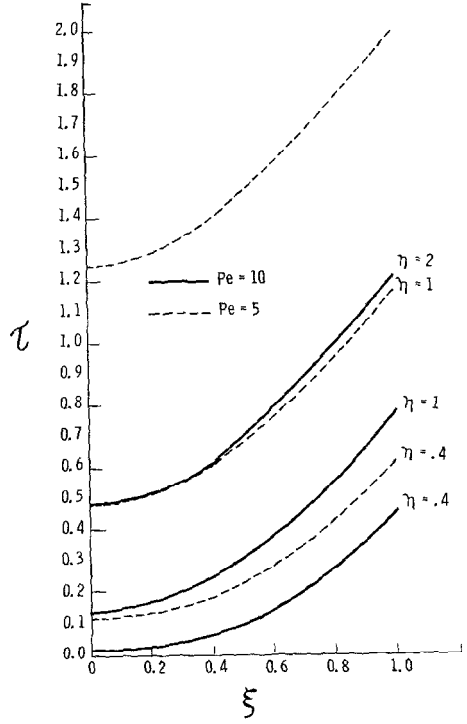

FIG. 2. Radial profiles of dimensionless temperatures for various downstream positions; $Pe = 5, 10$.

for 0.5% accuracy over most of the tube. The quantity F_1 is given in Table 4 for $N = 4$, 12, 18, 21 and 25. The tube entrance condition, Eq. (15), appears to be approximated very well, in the least-squares sense, for $N \geqslant 12$.

Comparing Tables III and IV, the quantity F_1 appears to be a more sensitive test for convergence than $Nu(\eta)$, since the negative exponential factors in Eq. (21) speed the convergence rate of the latter.

TABLE V

Expansion Coefficients Obtained by Regular Collocation Method; $Pe = 10$ and $N = 21$

n	C_n (Regular Collocation)
1	0.474033
2	−0.218521
3	0.124736
4	−0.079866
5	0.055250
6	−0.041604
7	0.032103
8	−0.027073
9	0.021887
10	−0.020452
11	0.016240
12	−0.017547
13	0.011904
14	−0.018089
15	0.003814
16	−0.033615
17	−0.026748
18	0.037682
19	0.256795
20	−0.028286
21	−0.189712

The Nusselt function is plotted in Fig. 1 for $Pe = 5$ and 10 and $N = 21$. The corresponding values of τ vs. ξ are plotted for various η in Fig. 2. Apparently, the profiles approach similarity in shape as η increases.

B. *Collocation Method*

For comparison with the Galerkin method, the method of collocation was also used to approximate the coefficients C_n in Eq. (15) for $Pe = 10$. Equation (19) was used for $N = 21$ points ($\xi = 0, .05, 0.10, 0.15, 0.20, \ldots, 0.95, 1.00$); the results are given in Table V.

Although the function

$$F_2 = \sum_{n=1}^{21} C_n g_n - (\xi^4/4 - \xi^2 + 7/24) \tag{23}$$

vanishes at the points of collocation, it oscillates between fairly large positive and negative values between these points. In Fig. 3 a comparison is made between the functions F_2 computed by the methods of collocation and Galerkin when $N = 21$ and $Pe = 10$.

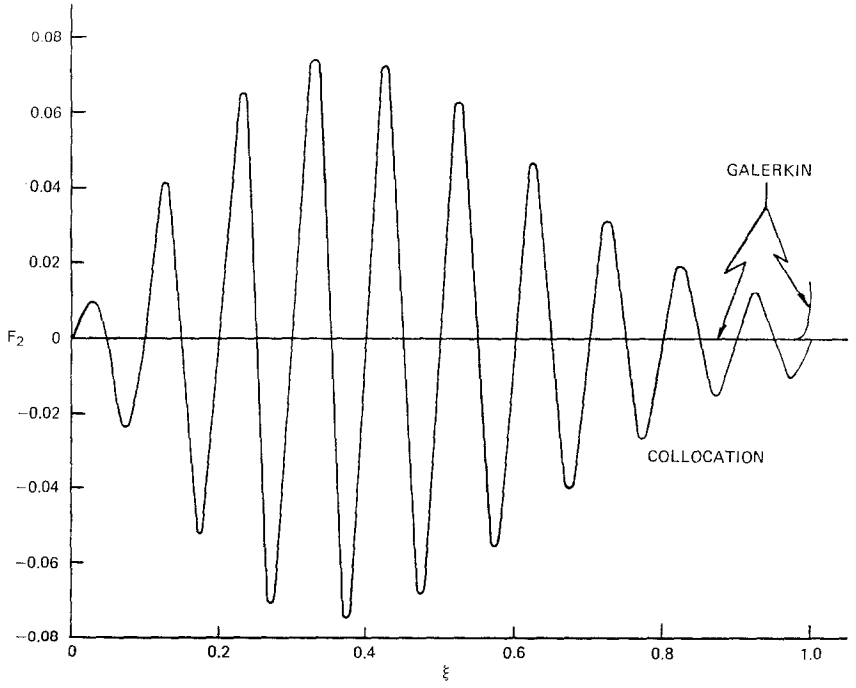

FIG. 3. Error of methods of collocation and Galerkin in fitting entrance condition.

CONCLUSIONS

The boundary method of Galerkin offers a computational advantage over the interior method of Galerkin since the latter requires solution of N systems of nonhomogeneous linear equations with dimension $(N-1)$ plus another N-dimensional system of nonhomogeneous linear equations. The former method requires only the solution of a single N-dimensional system of nonhomogeneous linear equations.

Furthermore, calculation of the ϕ_n and $R_n(\phi_n, \xi)$ using confluent hypergeometric functions requires considerably less computer time than the Runge–Kutta procedure.

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